ASSIGNMENT 3 — LAMBDA CALCULUS

COMP 3010 — ORGANIZATION OF PROGRAMMING LANGUAGES

1. Syntax

Remember that the abstract syntax of the λ-calculus is defined by the grammar

$$M ::= x \mid M\ M \mid \lambda x.M$$

where x, y, and z stand for *identifier names* (*a.k.a. variables*) used inside expressions in the language of the λ-calculus.

Ambiguities in the grammar, where multiple juxtapositions and λs are mixed, are resolved by these associativity and precedence rules:

$$M_1\ M_2\ \dots\ M_n = ((M_1\ M_2)\ \dots)\ M_n \qquad M_1\ \lambda x.M_2 = M_1\ (\lambda x.M_2)$$
$$\lambda x.M_1\ M_2\ \dots\ M_n = \lambda x.(M_1\ M_2\ \dots\ M_n) \qquad \lambda x_1.\lambda x_2.M = \lambda x_1.(\lambda x_2.M)$$

In other words, application (written as juxtaposition $M_1\ M_2$) associates to the *left*, abstraction (written as $\lambda x.M$) associates to the *right*, a λ to the right of an application (a λ argument $M_1\ (\lambda x.M_2)$) has a *higher precedence* than that application, and an application to the right of a λ (an applied function body $\lambda x.(M_1\ M_2)$) has a *higher precedence* than the λ.

Exercise 1 (Multiple Choice)**.** Which of the following, fully-parenthesized λ-calculus expressions stand for the same syntax tree as $y\ \lambda x.x\ y\ z$?

(a) $y\ (\lambda x.x)\ (y\ z)$
(b) $((y\ (\lambda x.x))\ y)\ z$
(c) $y\ (\lambda x.(x\ (y\ z)))$
(d) $y\ (\lambda x.((x\ y)\ z))$

Exercise 2 (Short Answer)**.** Use the above rules of precedence/associativity to draw the syntax tree represented by the fully-parenthesized version of $\lambda x.\lambda y.y\ z\ x$

2. Alpha

Exercise 3 (Short Answer)**.** Remember that the set of *free variables* of an expression M, written as $\mathrm{FV}(M)$, is *inductively defined* on the *syntax of* M like so:

$$\mathrm{FV}(x) = \{x\}$$
$$\mathrm{FV}(M_1\ M_2) = \mathrm{FV}(M_1) \cup \mathrm{FV}(M_2)$$
$$\mathrm{FV}(\lambda x.M) = \mathrm{FV}(M) - \{x\}$$

Give the set of *free variables* found in the expression $(\lambda x.\lambda y.y\ (\lambda z.x))\ (\lambda x.z)$:

$$\mathrm{FV}((\lambda x.\lambda y.y\ (\lambda z.x))\ (\lambda x.z)) = ?$$

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Exercise 4 (This or That)**.** Remember that the operation which *renames x to y in M*, written $[x/y]M$, is *inductively defined* by transforming the *syntax of M* as:

$$
\begin{aligned}
[y/x]x &= x \\
[y/x]z &= z && \text{(if } z \neq x\text{)} \\
[y/x](M_1\ M_2) &= ([y/x]M_1)\ ([y/x]M_2) \\
[y/x](\lambda x.M) &= \lambda x.M \\
[y/x](\lambda z.M) &= \lambda z.([y/x]M) && \text{(if } z \neq x\text{)}
\end{aligned}
$$

The law of α-equivalence is defined in terms of renaming as:

$$\lambda x.M =_\alpha \lambda y.([y/x]M) \qquad \text{(if } y \notin \mathrm{FV}(M)\text{)}$$

Determine which of the following are valid renamings. In other words, say whether or not each pair of λ-calculus expressions are equal according to α-equivalence.

(1) $\lambda x.(x\ x) =_\alpha \lambda y.(y\ y)$?
(2) $\lambda x.(\lambda x.x) =_\alpha \lambda y.(\lambda z.z)$?
(3) $\lambda x.(x\ y) =_\alpha \lambda y.(y\ y)$?
(4) $(\lambda x.x)\ x =_\alpha (\lambda y.y)\ x$?
(5) $(\lambda x.x)\ x =_\alpha (\lambda y.y)\ y$?

3. Beta

Exercise 5 (Multiple Choice)**.** Remember that the operation which *substitutes M' for x in M*, written $[M'/x]M$, is *inductively defined* by transforming the *syntax of M* like so:

$$
\begin{aligned}
[M'/x]x &= M' \\
[M'/x]z &= z && \text{(if } z \neq x\text{)} \\
[M'/x](M_1\ M_2) &= ([M'/x]M_1)\ ([M'/x]M_2) \\
[M'/x](\lambda x.M) &= \lambda x.M \\
[M'/x](\lambda z.M) &= \lambda z.([M'/x]M) && \text{(if } z \neq x \text{ and } z \notin \mathrm{FV}(M')\text{)}
\end{aligned}
$$

What is the result of the substitution $[(\lambda x.x\ x)/y](y\ (\lambda z.y\ z))$?

(1) $[(\lambda x.x\ x)/y](y\ (\lambda z.y\ z)) = (\lambda x.x\ x)\ (\lambda z.y\ z)$
(2) $[(\lambda x.x\ x)/y](y\ (\lambda z.y\ z)) = y\ (\lambda z.(\lambda x.x\ x)\ z)$
(3) $[(\lambda x.x\ x)/y](y\ (\lambda z.y\ z)) = (\lambda x.x\ x)\ (\lambda z.(\lambda x.x\ x)\ z)$
(4) $[(\lambda x.x\ x)/y](y\ (\lambda z.y\ z)) = (\lambda x.x\ x)\ (\lambda z.(\lambda x.x\ x)\ (\lambda x.x\ x))$

Exercise 6 (Multiple Choice)**.** Remember that the law of β-reduction is defined in terms of substitution as:

$$(\lambda x.M)\ M' \mapsto_\beta [M'/x]M$$

and the *call-by-name operational semantics* for the λ-calculus is defined by applying the above β-reduction rule inside of these evaluation contexts E defined like so:

$$E ::= \square \mid E\ M \qquad\qquad \frac{M \mapsto_\beta M'}{E[M] \mapsto_\beta E[M']}$$

Which of the following is the result of evaluating the expression

$$(\lambda x.\lambda y.\lambda z.y\ x\ z)\ (\lambda x.x)\ (\lambda x.\lambda y.x)\ z$$

according to the call-by-name operational semantics (as shown above)? In other words, which of the following do you get by applying β-reduction (as shown above) as many times as possible in a sequence like so:

$$(\lambda x.\lambda y.\lambda z.y\ x\ z)\ (\lambda x.x)\ (\lambda x.\lambda y.x)\ z \mapsto M_1 \mapsto M_2 \mapsto \cdots \mapsto \text{answer}$$

(a) $(\lambda x.x)$
(b) $(\lambda x.\lambda y.x)$
(c) $(\lambda y.z)$
(d) z

Exercise 7 (Short Answer)**.** What happens when you evaluate the λ-calculus expression $(\lambda x.x\ x)\ (\lambda x.x\ x)$? by repeatedly applying β-reduction? If β-reduction eventually stops, what final expression is the result? If you think β-reduction does not stop, explain why not?

4. Eta

Exercise 8 (This or That)**.** Remember that the law of η-reduction is defined in terms of free variables like so:

$$(\lambda x.(M\ x)) \to_\eta M \qquad\qquad (\text{if } x \notin \mathrm{FV}(M))$$

Say which of the following are correct η-reductions (according to the above rule) and which are not.

(1) $\lambda x.x \to_\eta x$
(2) $\lambda y.(x\ y) \to_\eta x$
(3) $\lambda x.(x\ x) \to_\eta x$
(4) $\lambda x.((\lambda y.y)\ x) \to_\eta \lambda y.y$
(5) $\lambda x.((y\ x)\ x) \to_\eta y\ x$